Construction of Normal Fuzzy Numbers Using the Mathematics of Partial Presence

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Abstract-**Two independent laws of measure theoretically defined randomness, one in the interval [α, β] and the other in the interval [β, γ], are necessary and sufficient to define a normal fuzzy number [α, β, γ], the membership function of which is the distribution function with reference to the law of randomness defined in [α, β] and is the complementary distribution function with reference to the law of randomness defined in [β, γ]. Thus to construct a normal fuzzy number [α, β, γ], we need to define two probability measures, one in [α, β] and the other in [β, γ].**

Keywords-Randomness; Fuzziness; Probability Measure; Order Statistics; the Glivenko – Cantelli Theorem; Superimposition of Sets

I. INTRODUCTION

In the theory of fuzzy sets, the membership function of a normal fuzzy number is defined as one that shows the level of partial presence of any element in a given interval. How exactly one has to construct a normal fuzzy number has however been left unanswered. Indeed, it has been continually believed right from the beginning that the membership function of a normal fuzzy number would in some way lead to a probability law defined in the same interval in which the fuzzy number is defined. All sorts of heuristic results towards linking fuzziness with probability have been published over the world in the last half century based on an illogical belief that a probability law can be defined in the same interval in which a law of fuzziness has been defined, and either of these two laws could possibly be deduced from the other. About such deductions, the less said the better. The workers have extended earlier results without bothering to understand possibly what they are extending. It is also true that one can understand anything only if it is based on logic; to understand mathematics based on belief is not possible anyway. We are hereby nullifying all such formalisms misinterpreting measure theory in general and probability theory in particular, established so far in the name of the mathematics of fuzziness.

The present author has shown that every normal law of fuzziness is rooted at two independent laws of randomness [1, 2, 3]. On the basis of that, it was shown how a normal fuzzy number can be constructed [4]. We have meanwhile come to understand that the Glivenko – Cantelli Theorem on uniform convergence of empirical probability distribution functions, which was used in our proof, is one thing that is not quite known to most of the people outside the statistics fraternity. We have further observed that the measure theoretic definition of randomness is something that most of the people have not properly looked into. Finally, we have been made to understand that our set operation of superimposition [1-4] too needs some more detailed exposition.

In this context, we would like to mention that in the Zadehian definition, a fuzzy set and its complement do not complement each other. After all, a statement and its complement can never have anything common. Indeed, defining the complement in the currently followed manner had been a great mathematical blunder. We have discussed this matter elsewhere [3].

In this article, we are going to explain the matters regarding the construction of a normal fuzzy number in detail. Just as a probability law can be constructed from sample observations, a normal law of fuzziness also can actually be constructed from sample observations. The form of the membership function of a normal fuzzy number need not be based on any subjective judgment. We need to know that the root of fuzziness is nothing but randomness. The rest can be left to be managed by the existing mathematics of statistics. We need to further know that fuzziness arises when a variable, daily temperature in any place for example, is of the interval type with the minimum and the maximum of the variable following some laws of randomness.

In what follows, we shall first discuss the concept of Order Statistics, and then we shall describe the Glivenko – Cantelli theorem on uniform convergence of an empirical probability distribution function to the underlying theoretical probability distribution function (see example [5], page 20) in a simple way. We would further discuss in detail the concept of superimposition of sets with reference to the mathematics of partial presence [2], an idea that was first established by the present author many years back [6]. We shall not cite any work on fuzziness authored by anyone else as citing blunders is of no use.

II. ORDER STATISTICS AND THE GLIVENKO-CANTELLI THEOREM

In every branch of study, certain results form the pillars of knowledge. Some of such results in course of time become classical, and may remain unknown to people who are not from that particular field of study. The Glivenko – Cantelli Theorem[1] is one such result that may not be well known to most of the people outside the statistics fraternity. This theorem describes the asymptotic behaviour of an empirical probability distribution function as the number of independent and identically distributed observations become countably infinite. This theorem which can be very well referred to as the basis of mathematical statistics, states that a data-based empirical distribution function converges uniformly to the underlying theoretical distribution function. Before explaining the implications of this theorem, we need to discuss first about order statistics.

Let X_1, X_2, … , X_n denote a random sample from a population with continuous cumulative distribution function F_X ([7], page 25). Since F_X is assumed to be continuous, the probability of any two or more of these random variables

[1] Named after Valery Ivanovich Glivenko and Fransesco Paolo Cantelli.

assuming equal magnitudes is zero. Therefore, there exists a unique ordered arrangement within the sample. At this point and hereafter we would like the readers to note that X_1, $X_2, \ldots, X_n$ may be ordered already in the sense that there is no probability law of errors governing their occurrences. They may occur following some law of physics for example, and even then what follows would be mathematically true.

Let $X_{(1)}$ denote the smallest of $X_1, X_2, \ldots, X_n$; $X_{(2)}$ denote the next smallest, etc.; and finally $X_{(n)}$ denote the largest. Then

$$X_{(1)} < X_{(2)} < \ldots < X_{(n)}$$

denotes the original sample after arrangement in the ascending order of magnitude. $X_{(r)}$ here is called the r-th order statistics. Let now the random variable X have the cumulative distribution function F_X. If F_X is continuous, the random variable

$$y = F_X(x)$$

is uniformly distributed over the interval (0, 1). Indeed, because

$$0 \le F_X(x) \le 1$$

for all x, we have

$$F_Y(y) = 0 \text{ if } y \le 0, \quad = 1 \text{ if } y \ge 1.$$

For $0 < y < 1$, writing $F_X(u) = y$, where u is the largest number satisfying this, it can be seen that $F_X(x) \le y$ is necessary as well as sufficient for $X \le u$. Accordingly,

$$F_Y(y) = F_X(u) = y,$$

thus leading us to the uniform law of randomness.

A consequence of this assertion is as follows. If X_1, $X_2, \ldots, X_n$ are random with the continuous distribution function F_X, then

$$F_X(X_1),\ F_X(X_2), \ldots, F_X(X_n)$$

are uniformly random. In the same way, for

$$X_{(1)} < X_{(2)} < \ldots < X_{(n)}$$

also

$$F_X(X_{(1)}), F_X(X_{(2)}), \ldots, F_X(X_{(n)})$$

would be uniformly random.

We now proceed to discuss what is known as an empirical distribution function. An empirical distribution function may be considered as an estimate of the cumulative distribution function defining the randomness concerned. For a sample of size n, this function $S_n(x)$ is defined as the proportion of values that do not exceed x. Accordingly, if $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics of a random sample, its empirical distribution function would be given by

$$S_n(x) = 0, \text{ if } x < X_{(1)},$$
$$= k / n, \text{ if } X_{(k)} \le x < X_{(k+1)}, k = 1, 2, \ldots, (n-1),$$
$$= 1, \text{ if } x \ge X_{(n)}.$$

X here being random, so would be $S_n(X)$. Writing

$$\Delta_i(t) = 0, \text{ if } X_i > t, \quad = 1, \text{ otherwise,}$$

we see that

$$S_n(x) = \sum \Delta_i(x) / n.$$

Therefore $nS_n(x)$ will have the law followed by the sum of n independent Bernoulli random variables $\Delta_i(x)$. Indeed in such a case, we would have

$$\text{Prob}[S_n(x) = k/n] = {}^nC_k [F_X(x)]^k [1 - F_X(x)]^{n-k}$$

for k = 0, 1, …, n. Hence the mathematical expectation of $S_n(x)$ would be given by

$$E[S_n(x)] = F_X(x).$$

In other words, $S_n(x)$ converges uniformly to $F_X(x)$ almost surely.

This leads to the Glivenko – Cantelli theorem [8] that can be stated as follows. "The limiting value of the supremum of the difference between $S_n(x)$ and $F_X(x)$, as n becomes infinitely large, converges to zero almost surely."

This theorem which would be of utmost importance in our efforts to prove that fuzzy sets do conform to measure theoretic norms, has been mentioned as a class room exercise in a text book by Ash and Doleans-Dade ([9], pages 331 – 332). We are now going to quote that exercise to show the simplicity of the theorem.

Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed probabilistic variables with a common probability distribution function F. Let

$$F_n(x, \omega), x \in R, \omega \in \Omega,$$

be the empirical distribution function of the X_i, based on n trials. Then for large n, $F_n(x, \omega)$ approximates F (x). For example, for n = 3, $X_1(\omega) = 2$, $X_2(\omega) = ½$, $X_3(\omega) = 7$,

$$F_3(x, \omega) = 0, \text{ if } x < ½,$$
$$= 1/3, \text{ if } ½ \le x < 2,$$
$$= 2/3, \text{ if } 2 \le x < 7,$$
$$= 1, \text{ if } x \ge 7.$$

Indeed, as we can see from this example, the values of k / n are not appearing uniformly, in the sense that the intervals [½, 2), [2, 7) etc. are not of equal length. Therefore the values of k / n would be geared to the law of randomness followed by the variable. Then the values of k / n would converge almost surely to the concerned law of randomness. This is the essence of the Glivenko – Cantelli theorem.

In our context, this theorem plays an important role[10]. Indeed the physical significance of this theorem is very straightforward. Suppose, there is a cumulative relative frequency distribution with six class intervals as follows. We can see the intervals are not necessarily of equal width. Had they been of equal width, we could say straightaway that the underlying distribution governing the population in this sample is uniform. So for unequal widths of intervals, if the frequencies are all equal giving us equal jumps in every subsequent intervals, the relative cumulative frequency distribution must anyway lead to the underlying probability distribution.

Class intervals	Cumulative Frequencies	Cumulative Relative Frequencies
$a_{(1)} - a_{(2)}$	1	1/6
$a_{(2)} - a_{(3)}$	2	2/6
$a_{(3)} - a_{(4)}$	3	3/6
$a_{(4)} - a_{(5)}$	4	4/6
$a_{(5)} - a_{(6)}$	5	5/6
$> a_{(6)}$	6	1

In what follows, we shall now discuss the mathematics of partial presence of an element in an interval. We would first need to explain an operation that we had named superimposition of sets. Use of the Glivenko–Cantelli theorem thereafter would lead us to our goal of defining the membership function of a normal fuzzy number in terms of two independent laws of randomness.

III. THE MATHEMATICS OF PARTIAL PRESENCE

Defined by the present author [6, 11], and later used in recognizing periodic patterns [12], the operation of set superimposition is expressed as follows. If the set A is superimposed over the set B, we get

$$A\,(S)\,B = (A-B) \cup (A \cap B)^{(2)} \cup (B-A)$$

where S represents the operation of superimposition, and $(A \cap B)^{(2)}$ represents the elements of $(A \cap B)$ occurring twice, provided that $(A \cap B)$ is not void. We have defined this operation keeping in view the fact that if two line segments A and B of unequal lengths are overdrawn one over the other, this is what we are going to see. It can be seen that for two intervals $A = [a_1, b_1]$ and $B = [a_2, b_2]$, we should have

$$[a_1, b_1]\,(S)\,[a_2, b_2]$$

$$= [a_1, a_2] \cup [a_2, b_1]^{(2)} \cup [b_1, b_2], \text{ if } a_1 < a_2 < b_1 < b_2,$$

$$= [a_1, a_2] \cup [a_2, b_2]^{(2)} \cup [b_2, b_1], \text{ if } a_1 < a_2 < b_2 < b_1,$$

$$= [a_2, a_1] \cup [a_1, b_1]^{(2)} \cup [b_1, b_2], \text{ if } a_2 < a_1 < b_1 < b_2,$$

$$= [a_2, a_1] \cup [a_1, b_2]^{(2)} \cup [b_2, b_1], \text{ if } a_2 < a_1 < b_2 < b_1,$$

where

i) $a_1 < a_2 < b_1 < b_2$,
ii) $a_1 < a_2 < b_2 < b_1$,
iii) $a_2 < a_1 < b_1 < b_2$, and
iv) $a_2 < a_1 < b_2 < b_1$

are the four different possibilities in this case. Here we have assumed without lossing any generality that $[a_1, b_1] \cap [a_2, b_2]$ is not void, or in other words

$$\max(a_i) \le \min(b_i),\ i = 1, 2.$$

A closer look would reveal that we can express this as follows. Indeed

$$[a_1, b_1]\,(S)\,[a_2, b_2] = [a_{(1)}, a_{(2)}] \cup [a_{(2)}, b_{(1)}]^{(2)} \cup [b_{(1)}, b_{(2)}]$$

where

i) $a_{(1)} = \min(a_1, a_2)$,
ii) $a_{(2)} = \max(a_1, a_2)$,
iii) $b_{(1)} = \min(b_1, b_2)$, and
iv) $b_{(2)} = \max(b_1, b_2)$.

This conversion in terms of ordered values is to be noted properly. We would soon see the applicability of this conversion in our discussions.

As an extension of this, for three intervals $A = [a_1, b_1]$, $B = [a_2, b_2]$ and $C = [a_3, b_3]$, such that $[a_1, b_1] \cap [a_2, b_2] \cap [a_3, b_3]$ is not void, or

$$\max(a_i) \le \min(b_i),\ i = 1, 2, 3,$$

we would have the following 36 identities

$$[a_1, b_1]\,(S)\,[a_2, b_2]\,(S)\,[a_3, b_3]$$

$= [a_1, a_2] \cup [a_2, a_3]^{(2)} \cup [a_3, b_1]^{(3)} \cup [b_1, b_2]^{(2)} \cup [b_2, b_3]$, if $a_1 < a_2 < a_3 < b_1 < b_2 < b_3$,

$= [a_1, a_3] \cup [a_3, a_2]^{(2)} \cup [a_2, b_1]^{(3)} \cup [b_1, b_2]^{(2)} \cup [b_2, b_3]$, if $a_1 < a_3 < a_2 < b_1 < b_2 < b_3$,

$= [a_2, a_1] \cup [a_1, a_3]^{(2)} \cup [a_3, b_1]^{(3)} \cup [b_1, b_2]^{(2)} \cup [b_2, b_3]$, if $a_2 < a_1 < a_3 < b_1 < b_2 < b_3$,

$= [a_2, a_3] \cup [a_3, a_1]^{(2)} \cup [a_1, b_1]^{(3)} \cup [b_1, b_2]^{(2)} \cup [b_2, b_3]$, if $a_2 < a_3 < a_1 < b_1 < b_2 < b_3$,

$= [a_3, a_1] \cup [a_1, a_2]^{(2)} \cup [a_2, b_1]^{(3)} \cup [b_1, b_2]^{(2)} \cup [b_2, b_3]$, if $a_3 < a_1 < a_2 < b_1 < b_2 < b_3$,

$= [a_3, a_2] \cup [a_2, a_1]^{(2)} \cup [a_1, b_1]^{(3)} \cup [b_1, b_2]^{(2)} \cup [b_2, b_3]$, if $a_3 < a_2 < a_1 < b_1 < b_2 < b_3$,

$= [a_1, a_2] \cup [a_2, a_3]^{(2)} \cup [a_3, b_1]^{(3)} \cup [b_1, b_3]^{(2)} \cup [b_3, b_2]$, if $a_1 < a_2 < a_3 < b_1 < b_3 < b_2$,

$= [a_1, a_3] \cup [a_3, a_2]^{(2)} \cup [a_2, b_1]^{(3)} \cup [b_1, b_3]^{(2)} \cup [b_3, b_2]$, if $a_1 < a_3 < a_2 < b_1 < b_3 < b_2$,

$= [a_2, a_1] \cup [a_1, a_3]^{(2)} \cup [a_3, b_1]^{(3)} \cup [b_1, b_3]^{(2)} \cup [b_3, b_2]$, if $a_2 < a_1 < a_3 < b_1 < b_3 < b_2$,

$= [a_2, a_3] \cup [a_3, a_1]^{(2)} \cup [a_1, b_1]^{(3)} \cup [b_1, b_3]^{(2)} \cup [b_3, b_2]$, if $a_2 < a_3 < a_1 < b_1 < b_3 < b_2$,

$= [a_3, a_1] \cup [a_1, a_2]^{(2)} \cup [a_2, b_1]^{(3)} \cup [b_1, b_3]^{(2)} \cup [b_3, b_2]$, if $a_3 < a_1 < a_2 < b_1 < b_3 < b_2$,

$= [a_3, a_2] \cup [a_2, a_1]^{(2)} \cup [a_1, b_1]^{(3)} \cup [b_1, b_3]^{(2)} \cup [b_3, b_2]$, if $a_3 < a_2 < a_1 < b_1 < b_3 < b_2$,

$= [a_1, a_2] \cup [a_2, a_3]^{(2)} \cup [a_3, b_2]^{(3)} \cup [b_2, b_1]^{(2)} \cup [b_1, b_3]$, if $a_1 < a_2 < a_3 < b_2 < b_1 < b_3$,

$= [a_1, a_3] \cup [a_3, a_2]^{(2)} \cup [a_2, b_2]^{(3)} \cup [b_2, b_1]^{(2)} \cup [b_1, b_3]$, if $a_1 < a_3 < a_2 < b_2 < b_1 < b_3$,

$= [a_2, a_1] \cup [a_1, a_3]^{(2)} \cup [a_3, b_2]^{(3)} \cup [b_2, b_1]^{(2)} \cup [b_1, b_3]$, if $a_2 < a_1 < a_3 < b_2 < b_1 < b_3$,

$= [a_2, a_3] \cup [a_3, a_1]^{(2)} \cup [a_1, b_2]^{(3)} \cup [b_2, b_1]^{(2)} \cup [b_1, b_3]$, if $a_2 < a_3 < a_1 < b_2 < b_1 < b_3$,

$= [a_3, a_1] \cup [a_1, a_2]^{(2)} \cup [a_2, b_2]^{(3)} \cup [b_2, b_1]^{(2)} \cup [b_1, b_3]$, if $a_3 < a_1 < a_2 < b_2 < b_1 < b_3$,

$= [a_3, a_2] \cup [a_2, a_1]^{(2)} \cup [a_1, b_2]^{(3)} \cup [b_2, b_1]^{(2)} \cup [b_1, b_3]$, if $a_3 < a_2 < a_1 < b_2 < b_1 < b_3$,

$= [a_1, a_2] \cup [a_2, a_3]^{(2)} \cup [a_3, b_2]^{(3)} \cup [b_2, b_3]^{(2)} \cup [b_3, b_1]$, if $a_1 < a_2 < a_3 < b_2 < b_3 < b_1$,

$= [a_1, a_3] \cup [a_3, a_2]^{(2)} \cup [a_2, b_2]^{(3)} \cup [b_2, b_3]^{(2)} \cup [b_3, b_1]$, if $a_1 < a_3 < a_2 < b_2 < b_3 < b_1$,

$= [a_2, a_1] \cup [a_1, a_3]^{(2)} \cup [a_3, b_2]^{(3)} \cup [b_2, b_3]^{(2)} \cup [b_3, b_1]$, if $a_2 < a_1 < a_3 < b_2 < b_3 < b_1$,

$= [a_2, a_3] \cup [a_3, a_1]^{(2)} \cup [a_1, b_2]^{(3)} \cup [b_2, b_3]^{(2)} \cup [b_3, b_1]$, if $a_2 < a_3 < a_1 < b_2 < b_3 < b_1$,

$= [a_3, a_1] \cup [a_1, a_2]^{(2)} \cup [a_2, b_2]^{(3)} \cup [b_2, b_3]^{(2)} \cup [b_3, b_1]$, if $a_3 < a_1 < a_2 < b_2 < b_3 < b_1$,

$= [a_3, a_2] \cup [a_2, a_1]^{(2)} \cup [a_1, b_2]^{(3)} \cup [b_2, b_3]^{(2)} \cup [b_3, b_1]$, if $a_3 < a_2 < a_1 < b_2 < b_3 < b_1$,

$= [a_1, a_2] \cup [a_2, a_3]^{(2)} \cup [a_3, b_3]^{(3)} \cup [b_3, b_1]^{(2)} \cup [b_1, b_2]$, if $a_1 < a_2 < a_3 < b_3 < b_1 < b_2$,

$= [a_1, a_3] \cup [a_3, a_2]^{(2)} \cup [a_2, b_3]^{(3)} \cup [b_3, b_1]^{(2)} \cup [b_1, b_2]$, if $a_1 < a_3 < a_2 < b_3 < b_1 < b_2$,

$= [a_2, a_1] \cup [a_1, a_3]^{(2)} \cup [a_3, b_3]^{(3)} \cup [b_3, b_1]^{(2)} \cup [b_1, b_2]$, if $a_2 < a_1 < a_3 < b_3 < b_1 < b_2$,

$= [a_2, a_3] \cup [a_3, a_1]^{(2)} \cup [a_1, b_3]^{(3)} \cup [b_3, b_1]^{(2)} \cup [b_1, b_2]$, if $a_2 < a_3 < a_1 < b_3 < b_1 < b_2$,

$= [a_3, a_1] \cup [a_1, a_2]^{(2)} \cup [a_2, b_3]^{(3)} \cup [b_3, b_1]^{(2)} \cup [b_1, b_2]$, if $a_3 < a_1 < a_2 < b_3 < b_1 < b_2$,

$= [a_3, a_2] \cup [a_2, a_1]^{(2)} \cup [a_1, b_3]^{(3)} \cup [b_3, b_1]^{(2)} \cup [b_1, b_2]$, if $a_3 < a_2 < a_1 < b_3 < b_1 < b_2$,

$= [a_1, a_2] \cup [a_2, a_3]^{(2)} \cup [a_3, b_3]^{(3)} \cup [b_3, b_2]^{(2)} \cup [b_2, b_1]$, if $a_1 < a_2 < a_3 < b_3 < b_2 < b_1$,

$= [a_1, a_3] \cup [a_3, a_2]^{(2)} \cup [a_2, b_3]^{(3)} \cup [b_3, b_2]^{(2)} \cup [b_2, b_1]$, if $a_1 < a_3 < a_2 < b_3 < b_2 < b_1$,

$= [a_2, a_1] \cup [a_1, a_3]^{(2)} \cup [a_3, b_3]^{(3)} \cup [b_3, b_2]^{(2)} \cup [b_2, b_1]$, if $a_2 < a_1 < a_3 < b_3 < b_2 < b_1$,

$= [a_2, a_3] \cup [a_3, a_1]^{(2)} \cup [a_1, b_3]^{(3)} \cup [b_3, b_2]^{(2)} \cup [b_2, b_1]$, if $a_2 < a_3 < a_1 < b_3 < b_2 < b_1$,

$= [a_3, a_1] \cup [a_1, a_2]^{(2)} \cup [a_2, b_3]^{(3)} \cup [b_3, b_2]^{(2)} \cup [b_2, b_1]$, if $a_3 < a_1 < a_2 < b_3 < b_2 < b_1$,

$= [a_3, a_2] \cup [a_2, a_1]^{(2)} \cup [a_1, b_3]^{(3)} \cup [b_3, b_2]^{(2)} \cup [b_2, b_1]$, if $a_3 < a_2 < a_1 < b_3 < b_2 < b_1$,

As earlier, it can be seen that

$$[a_1, b_1] \text{ (S) } [a_2, b_2] \text{ (S) } [a_3, b_3]$$

$$= [a_{(1)}, a_{(2)}] \cup [a_{(2)}, a_{(3)}]^{(2)} \cup [a_{(3)}, b_{(1)}]^{(3)} \cup [b_{(1)}, b_{(2)}]^{(2)} \cup [b_{(2)}, b_{(3)}],$$

where $a_{(1)}$, $a_{(2)}$, $a_{(3)}$ are values of a_1, a_2, a_3 arranged in increasing order of magnitude, and $b_{(1)}$, $b_{(2)}$, $b_{(3)}$ are values of b_1, b_2, b_3 arranged in increasing order of magnitude, where for example $[a_{(2)}, a_{(3)}]^{(2)}$ are elements of $[a_{(2)}, a_{(3)}]$ represented twice.

Using induction it can be shown that for n intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n],$$

subject to the condition that

$$[a_1, b_1] \cap [a_2, b_2] \cap \ldots \cap [a_{n-1}, b_{n-1}] \cap [a_n, b_n]$$

is not void, we would have $(n!)^2$ different cases that can in short be written as

$$[a_1, b_1] \text{ (S) } [a_2, b_2] \text{ (S) } \ldots \text{ (S) } [a_{n-1}, b_{n-1}] \text{ (S) } [a_n, b_n]$$

$$= [a_{(1)}, a_{(2)}] \cup [a_{(2)}, a_{(3)}]^{(2)} \cup \ldots \cup [a_{(n-1)}, a_{(n)}]^{(n-1)}$$

$$\cup [a_{(n)}, b_{(1)}]^{(n)} \cup [b_{(1)}, b_{(2)}]^{(n-1)} \cup \ldots$$

$$\cup [b_{(n-2)}, b_{(n-1)}]^{(2)} \cup [b_{(n-1)}, b_{(n)}],$$

where $a_{(1)}, a_{(2)}, \ldots, a_{(n)}$ are values of $a_1, a_2, \ldots, a_n$ arranged in increasing order of magnitude, and $b_{(1)}, b_{(2)}, \ldots, b_{(n)}$ also are values of $b_1, b_2, \ldots, b_n$ arranged in increasing order of magnitude, and for example $[a_{(n-1)}, a_{(n)}]^{(n-1)}$ are elements of $[a_{(n-1)}, a_{(n)}]$ represented (n-1) times. Observe that order statistical matters would now automatically enter into our discussions on application of set superimposition.

IV. AN APPLICATION OF SUPERIMPOSITION OF UNIFORMLY FUZZY SETS

We now proceed towards an application of the operation of set superimposition. We refer to the example of two line segments A and B of unequal lengths drawn one over the other again. Double representation creates a doubly dark situation in the common portion. Now if the level of darkness in the common portion is taken to be unity, then that in the other portions would have to be partial.

In other words, if $[a_1, b_1]^{(1/2)}$ and $[a_2, b_2]^{(1/2)}$ represent two uniformly fuzzy intervals both with membership value equal to ½ in the entire intervals, superimposition of the two equally fuzzy intervals $[a_1, b_1]^{(1/2)}$ and $[a_2, b_2]^{(1/2)}$ would give rise to

$$[a_1, b_1]^{(1/2)} \text{ (S) } [a_2, b_2]^{(1/2)}$$

$$= [a_{(1)}, a_{(2)}]^{(1/2)} \cup [a_{(2)}, b_{(1)}]^{(1)} \cup [b_{(1)}, b_{(2)}]^{(1/2)}.$$

This is like placing one translucent paper over another of equal opacity ½ to get the opacity doubled as a result in the common portion.

In the same way, if $[a_1, b_1]^{(1/3)}$, $[a_2, b_2]^{(1/3)}$ and $[a_3, b_3]^{(1/3)}$ represent three uniformly fuzzy intervals all with membership value equal to 1/3 in the entire intervals, superimposition of $[a_1, b_1]^{(1/3)}$, $[a_2, b_2]^{(1/3)}$ and $[a_3, b_3]^{(1/3}$ would give rise to

$$[a_1, b_1]^{(1/3)} \text{ (S) } [a_2, b_2]^{(1/3)} \text{ (S) } [a_3, b_3]^{(1/3)}$$

$$= [a_{(1)}, a_{(2)}]^{(1/3)} \cup [a_{(2)}, a_{(3)}]^{(2/3)} \cup [a_{(3)}, b_{(1)}]^{(1)} \cup [b_{(1)}, b_{(2)}]^{(2/3)} \cup [b_{(2)}, b_{(3)}]^{(1/3)}$$

So for n fuzzy intervals $[a_1, b_1]^{(1/n)}$, $[a_2, b_2]^{(1/n)}$, …, $[a_n, b_n]^{(1/n)}$ all with membership value equal to 1/n in the entire intervals, we shall have

$$[a_1, b_1]^{(1/n)} \text{ (S) } [a_2, b_2]^{(1/n)} \text{ (S) } \ldots \text{ (S) } [a_n, b_n]^{(1/n)}$$

$$= [a_{(1)}, a_{(2)}]^{(1/n)} \cup [a_{(2)}, a_{(3)}]^{(2/n)} \cup \ldots \cup [a_{(n-1)}, a_{(n)}]^{((n-1)/n)}$$

$$\cup [a_{(n)}, b_{(1)}]^{(1)} \cup [b_{(1)}, b_{(2)}]^{((n-1)/n)} \cup \ldots \cup [b_{(n-2)}, b_{(n-1)}]^{(2/n)} \cup [b_{(n-1)}, b_{(n)}]^{(1/n)},$$

where, for example, $[b_{(1)}, b_{(2)}]^{((n-1)/n)}$ represents the uniformly fuzzy interval $[b_{(1)}, b_{(2)}]$ with membership ((n-1) /n) in the entire interval, $a_{(1)}$, $a_{(2)}$, ………, $a_{(n)}$ being values of a_1, a_2, ………, a_n arranged in increasing order of magnitude, and $b_{(1)}$, $b_{(2)}$, ………, $b_{(n)}$ being values of b_1, b_2, ………, b_n arranged in increasing order of magnitude.

The idea can be seen to be valid from another standpoint too. We have exemplified a case of a cumulative relative frequency distribution with six class intervals as follows.

Class Intervals	Cumulative Levels of Presence
$> a_{(1)}$	1/6
$> a_{(2)}$	2/6
$> a_{(3)}$	3/6
$> a_{(4)}$	4/6
$> a_{(5)}$	5/6
$> a_{(6)}$	1

What we have done is that we had considered the intervals vertically. Now it can be seen that the cumulative levels of presence resemble the levels of fuzziness that is known as membership. Likewise, for the other part of the superimposed fuzzy intervals, we have the following:

Class Intervals	Complementary Cumulative Levels of Presence
$< b_{(1)}$	1
$< b_{(2)}$	5/6
$< b_{(3)}$	4/6
$< b_{(4)}$	3/6
$< b_{(5)}$	2/6
$< b_{(6)}$	1/6

V. CONSTRUCTION OF A NORMAL FUZZY NUMBER

We would like to say a few things about probability space and randomness first. The definition of probability spaces is geared to randomness in general and not to probability in particular. In fact, randomness is one term very widely misunderstood. It is generally said that a random variable is one that is associated with some probability law of errors. However, from the measure theoretic standpoint, if

$$\int_a^b f(x)\,dx = 1,$$

then X is said to be a random variable with reference to the density function f(x) with X defined in the interval [a, b]. In other words, if a variable is random in the measure theoretic sense, it need not be probabilistic in the statistical sense. For example,

$$\int_0^1 2x\,dx = 1,$$

and therefore X defined in [0,1] here is random by definition, but not necessarily probabilistic following some probability law of errors. We would like to cite here the book by Rohatgi and Saleh ([13], pages 41 - 43) who have clearly mentioned that the notion of probability does not enter into the definition of a random variable. Accordingly, all results of probability theory would be applicable to random variables with randomness defined in the measure theoretic sense. For example, the Glivenko – Cantelli theorem was framed keeping in view the notion of probability, and therefore the theorem would be true even when we define randomness in the measure theoretic sense.

Consider now two probability spaces (Ω_1, A_1, Π_1) and (Ω_2, A_2, Π_2) where Ω_1 and Ω_2 are real intervals $[\alpha, \beta]$ and $[\beta, \gamma]$ respectively. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be realizations in $[\alpha, \beta]$ and $[\beta, \gamma]$ respectively. Recall now the fuzzy intervals $[a_1, b_1]^{(1/n)}$, $[a_2, b_2]^{(1/n)}$, …, $[a_n, b_n]^{(1/n)}$, all with constant membership 1/n. The values of membership of the superimposed fuzzy intervals are

$$1/n, 2/n, \ldots, (n-1)/n, 1, (n-1)/n, \ldots, 2/n, 1/n.$$

We are now going to take one big step. Observe that the values of membership considered in two parts,

$$(0, 1/n, 2/n, \ldots, (n-1)/n, 1)$$

and

$$(1, (n-1)/n, \ldots, 2/n, 1/n, 0)$$

can actually define an empirical distribution and a complementary empirical distribution on $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ respectively. This particular shift from one paradigm to another, from fuzziness to randomness, would be very significant towards understanding our formalisms. For realizations of the values of $a_{(1)}, a_{(2)}, \ldots, a_{(n)}$ in increasing order and of $b_{(1)}, b_{(2)}, \ldots, b_{(n)}$ again in increasing order, it can be seen that if we define

$$\begin{aligned}\Pi_1(x) &= 0, \text{ if } x < a_{(1)},\\ &= (r-1)/n, \text{ if } a_{(r-1)} \le x \le a_{(r)}, r = 2, 3, \ldots, n,\\ &= 1, \text{ if } x \ge a_{(n)},\\ \Pi_2(x) &= 1, \text{ if } x < b_{(1)},\\ &= 1 - (r-1)/n, \text{ if } b_{(r-1)} \le x \le b_{(r)}, r = 2, 3, \ldots, n,\\ &= 0, \text{ if } x \ge a_{(n)},\end{aligned}$$

then we must have as n becomes countably infinite

$$\Pi_1(x) \to \Psi_1(x), \alpha \le x \le \beta,$$

$$\Pi_2(x) \to (1 - \Psi_2(x)), \beta \le x \le \gamma.$$

where $\Psi_1(x)$, $\alpha \le x \le \beta$, and $(1 - \Psi_2(x))$, $\beta \le x \le \gamma$, are the underlying theoretical distribution functions.

At this point, we note the equivalence between the definitions of the Dubois – Prade left reference function of a normal fuzzy number and a distribution function, and the equivalence between the definitions of the Dubois – Prade right reference function of that normal fuzzy number and a complementary distribution function. It is obvious that for every distribution function, there would be a density function. Hence for the distribution functions $\Psi_1(x)$ and $(1- \Psi_2(x))$ defined on $\alpha \le x \le \beta$ and $\beta \le x \le \gamma$ respectively, densities $d\Psi_1(x)/dx$ and $d(1- \Psi_2(x))/dx$ would anyway exist.

Thus it has been noticed that to construct a normal fuzzy number $N = [\alpha, \beta, \gamma]$, it is necessary as well as sufficient to define two densities $d\Psi_1(x)/dx$ and $d(1- \Psi_2(x))/dx$ in the intervals $[\alpha, \beta]$ and $[\beta, \gamma]$ respectively, so that $\Psi_1(x)$ and $\Psi_2(x)$ can give us the membership function of N. We can summarize our findings in the form of the following theorem.

Theorem: Two densities $d\Psi_1(x)/dx$ and $d(1- \Psi_2(x))/dx$ in the intervals $[\alpha, \beta]$ and $[\beta, \gamma]$ respectively can define a normal fuzzy number $N = [\alpha, \beta, \gamma]$ with membership function

$$\begin{aligned}\mu_N(x) &= \Psi_1(x), \text{ if } \alpha \le x \le \beta,\\ &= \Psi_2(x), \text{ if } \beta \le x \le \gamma,\\ &= 0, \text{ otherwise.}\end{aligned}$$

Assume now that we have data of maximum and minimum temperature collected at any particular place daily for a large number of days. The probability laws of error followed by the random variables, the minimum temperature and the maximum temperature, can be found out using the Glivenko – Cantelli theorem, and we can construct a normal fuzzy number $[\alpha, \beta, \gamma]$ with the intervals $[\alpha, \beta]$ and $[\beta, \gamma]$ being the sample spaces for minimum and maximum temperature respectively. In this case, randomness has to be associated to probability laws of errors, a detailed discussion of which are included in [2]. However, when we see different shades of darkness in a room, the fuzziness concerned can be explained in our way again, this time the randomness concerned being not associated to any probability law of errors.

As an application of the theorem stated above, we illustrate the following example. For the uniform density function

$$f(x) = 1/(\beta - \alpha), \alpha \le x \le \beta,$$

the distribution function is given by

$$\begin{aligned}F(x) &= \int_\alpha^x f(x)\,dx\\ &= (x - \alpha)/(\beta - \alpha).\end{aligned}$$

Similarly, for the uniform density function

$$g(x) = 1/ (\gamma - \beta),\ \beta \le x \le \gamma,$$

the distribution function is given by

$$G(x) = (x - \beta)/ (\gamma - \beta).$$

It can be seen that F(x) here is the left reference function and (1 – G(x)) here is the right reference function as defined by Dubois and Prade of the triangular number [α, β, γ] with membership

$$\begin{aligned} \mu(x) &= (x - \alpha)/ (\beta - \alpha),\ \text{if } \alpha \le x \le \beta, \\ &= 1 - (x - \beta)/ (\gamma - \beta),\ \text{if } \beta \le x \le \gamma, \\ &= 0,\ \text{otherwise.} \end{aligned}$$

Thus we have seen that assumption of existence of two uniform densities, the simplest form of all densities, in [α, β] and [β, γ], is sufficient for the construction of a triangular fuzzy number [α, β, γ], the simplest form of all fuzzy numbers. Other kinds of densities would be sufficient accordingly to construct other kinds of fuzzy numbers.

VI. CONCLUSIONS AND DISCUSSIONS

For a normal fuzzy number of the type N = [α, β, γ] with membership function

$$\begin{aligned} \mu_N(x) &= \Psi_1(x),\ \text{if } \alpha \le x \le \beta, \\ &= \Psi_2(x),\ \text{if } \beta \le x \le \gamma, \text{ and} \\ &= 0,\ \text{otherwise,} \end{aligned}$$

with $\Psi_1(\alpha) = \Psi_2(\gamma) = 0$, $\Psi_1(\beta) = \Psi_2(\beta) = 1$, the partial presence of a value x of the variable X in the interval [α, γ] is expressible as

$$\mu_N(x) = \theta\ \Psi_1(x) + (1 - \theta)\ \Psi_2(x),$$

with

$$\begin{aligned} \theta &= 1,\ \text{if } \alpha \le x \le \beta, \text{ and} \\ &= 0,\ \text{if } \beta \le x \le \gamma. \end{aligned}$$

This indeed is the randomness-fuzziness consistency principle that people had been searching for the last half century. We are not proposing this principle; we have actually established it. We assure the readers that there simply cannot be a second such principle linking fuzziness and randomness. While deducing it, we have not followed the leader. We have been able to explain exactly wherefrom the Dubois-Prade left and right functions originate. The two Dubios – Prade functions do not just fall from the blue; they are firmly rooted at two independent laws of randomness.

We can conclude that given two intervals [α, β] and [β, γ], two probability measures defined on these two intervals can define a normal fuzzy number [α, β, γ]. The distribution function in the interval [α, β] and the complementary distribution function in the interval [β, γ] can together form the membership function of the fuzzy number. This is how a normal fuzzy number has to be constructed. There cannot be any other alternative.

All results available in the literature towards linking probability and fuzziness were aimed at establishing one probability law from any given normal law of fuzziness. The existing probability – possibility consistency principles were all framed keeping that in mind. As we have seen, constructing one single probability law from a normal law of fuzziness is not possible, and hence those probability–possibility consistency principles must necessarily be wrong.

The measure theoretic matters with reference to a fuzzy number must be geared to the classical probability measure. The mathematics done in the name of defining a fuzzy measure which is in no way a measure in the classical sense is not logical. In defining the so called fuzzy measure, logic was forced to follow mathematics. Mathematics should follow logic; it must never be the other way around. A pair of probability measures can define normal fuzziness. We do not need to define a fuzzy measure.

In the entire literature in statistics, there is no reference to finding the area under a probability distribution function. When we integrate a function, the result must necessarily have some physical significance. Integrating a distribution function is of no meaning. If someone still goes ahead towards defining something like that, and thereafter if some weird logic is imposed upon that kind of findings, then that cannot be called mathematics at least, whatever else that may be. Measure theory does not allow us to do a thing like that.

Just as trying to find the area under the distribution function of a random variable does not have any meaning, trying to find the area under the membership function too cannot have any meaning. Accordingly, all results established by integrating the membership function, or by integrating some function of the membership function, must be rejected forthwith. In the process, measure theory in general and probability theory in particular have been misinterpreted too far. Such misinterpretations should not continue any further.

Defining a fuzzy number in our way is helpful in explaining fuzzy arithmetic in a much simpler way. Of course, the method of α–cuts is available in the literature on fuzziness, and this method is sufficient to do fuzzy arithmetic. Our method too can be used to do the same. The method of α–cuts fail in situations when the concerned equation is not reversible. Our method fails too in this situation. The defect is not in the methods. After all, not all equations are solvable theoretically, and therefore an equation of the type y = f(x) cannot always be explicitly written as x = g(y). This is why the two methods fail together sometimes.

Beyond the method of α–cuts, most of the results established in the mathematics of fuzziness so far are incorrect. They are incorrect because they are based on entirely wrong assumptions. However in the method of α – cuts, for any value of α a line parallel to the X–axis where the fuzzy number interval is defined, would cut the membership curve at two points. It will cut at the same two points whether the membership function is defined as one function over the entire interval or as two different functions in two mutually exclusive and exhaustive parts of the same interval. That is why the mathematics of using the method of α–cuts is absolutely correct.

Extension of the earlier results will however continue unabated. It would be difficult for the workers to come out of their cult for some time at least. In fact, most of the workers might never agree that what we have established in this regard is the truth, the logical truth. We have therefore decided that instead of crying hoarse that most of the mathematics of fuzziness is illogical, we should rather go for a new nomenclature for this kind of uncertainty. We have since gone forward to do so, and have already started to call a fuzzy set an imprecise set [14]. Now that we are openly declaring that most of the current researches on theory and applications of fuzziness are wrong, we may perhaps be ridiculed. Once upon

a time, millions believed that our solar system is geocentric. The reality had always however been entirely different. In every strife between scientific truth demonstrated by one single individual and illogical belief supported by millions, truth had always been the instantaneous loser. But finally truth always prevails. We are facing a similar situation. It is not going to be easy for us single-handedly to make the workers understand that what they have been doing is not mathematics. We hope, the situation will change soon.

We had tried to disseminate these findings internationally in the mid-nineties of the last century. We were not allowed to do so by people who were mere followers of certain illogical beliefs, and they were neither able to see nor interested in knowing the reason. We then decided to publish the findings locally to keep our claims alive [6]. Only after the application of the operation of set superimposition led to a publication in the realm of computer science [12], we decided to reappear in the scene. The workers would hopefully see reason now and start the process of rectification of the mathematics of fuzziness before it is too late.

The origin of the mathematics of fuzziness had been a great event in the history of mathematics. The growth unfortunately had been malignant, and therefore the developments have become rather unearthly. Unless properly operated upon fast, the theory of fuzzy sets is certain to die in course of time. In fact, it is rather strange that it could survive so long.

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